SOME RESULTS ON EXTENDED INVERSE PROBLEM OF $A + 2 \cdot A$

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Abstract. Let A be a finite set of integers and $A + 2 \cdot A = \{a + 2a' : a, a' \in A\}$. An extended inverse problem associated with the sumset $A + 2 \cdot A$ is to determine the underlying set A when the size of the sumset $A + 2 \cdot A$ deviates from the minimum possible size. In this paper, we study some extended inverse problems for $A + 2 \cdot A$. Further, we find all the possible arithmetic structures of A for certain cardinalities of $A + 2 \cdot A$ and use them to address extended inverse problems in the Baumslag–Solitar group $BS(1, 2)$.

1. Introduction

In an additive abelian group G, the *sumset* of subsets A and B of G is denoted by $A + B$ and is defined by

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

For a positive real number r, the *dilation* of A by r, denoted by $r \cdot A$, is the set $r \cdot A = \{ra : a \in A\}$. Sumsets of the type $r_1 \cdot A + \cdots + r_k \cdot A$ commonly appear in combinatorial number theory. This sumset of dilated sets has drawn recent attraction in various contexts such as the sum-product problems in finite fields, the upper and lower bound problems for n-ary linear forms, etc., with notable research carried out by several authors. For detailed work, one may refer to [2–5, 7–11]. In particular, they studied sums involving two dilates of the form

$$A + r \cdot A = \{a + rb : a, b \in A\}.$$

A fundamental problem in additive number theory involves investigating the structure of the sumset $A + r \cdot A$ when the set A is given. This problem is called the *direct problem*. An *inverse problem* involves the study of the structure of the set A based on the properties of its sumset $A + r \cdot A$.

A significant number of direct and inverse problems involving $A + r \cdot A$ have been studied. The direct and inverse problems for the sumset $A + 2 \cdot A$, where A is a finite set of integers, were studied by Cilleruelo et al. [4]. Inverse problems of this nature, where the precise bound is presumed, are the ordinary inverse problems. The phrase *extended inverse problem* is used to describe inverse problems in which a small deviation from the exact bound is permissible. Since the cardinality of such sumsets of the set A is dilation and translation invariant of the set A, we assume that $\min(A) = 0$ and the greatest common divisor of the set A, denoted by $d(A)$, is equal to 1. Freiman et al. [5] proved the following extended inverse theorem for $A + 2 \cdot A$.

Theorem 1.1 ([5]). *Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of k (≥ 1) integers with $a_0 < a_1 < \cdots < a_{k-1}$. Then the following statements hold.*

(i) *If $1 \leq k \leq 2$, then $|A + 2 \cdot A| = 3k - 2$ and A is an arithmetic progression.*

(ii) *If $k \geq 3$ and*
$$|A + 2 \cdot A| = 3k - 2 + h < 4k - 4, \text{ where } h \geq 0,$$
then A is a subset of arithmetic progression $P = \{a_0, a_0 + d, a_0 + 2d, \ldots, a_0 + (l-1)d\}$ such that
$$|P| \leq k + h = |A + 2 \cdot A| - 2k + 2 \leq 2k - 3.$$

(iii) *If $k \geq 1$ and $|A + 2 \cdot A| = 3k - 2$, then A is an arithmetic progression.*

This article addresses some extended inverse problems related to $A + 2 \cdot A$, where A is a finite set of integers. We determine the structure of the set A when the cardinality of $A + 2 \cdot A$ exceeds its optimal lower bound, $3|A| - 2$, by a small amount. More precisely, we find the structure of A in each of the cases $|A + 2 \cdot A| = 3|A| - 1$, $|A + 2 \cdot A| = 3|A|$, and $|A + 2 \cdot A| = 3|A| + 1$. Note that these cases are obtained by setting $h = 1, 2$, and 3, respectively, in Theorem 1.1 (ii). Hence, in these cases, we have $k \geq 4$, $k \geq 5$, and $k \geq 6$, respectively. Initially, we determine the potential structure of A for specific cardinalities of $A + 2 \cdot A$. Subsequently, we find the precise structure of A.

The following theorems are the main results related to sums of dilates discussed in this article.

Theorem 1.2. *Let A be a set of four integers such that $|A + 2 \cdot A| = 11$. Then $A = \{0, 2, 3, 4\}$ or $\{0, 1, 2, 4\}$.*

Theorem 1.3. *There is no set A of k (≥ 5) integers such that $|A + 2 \cdot A| = 3k - 1$.*

Theorem 1.4. *Let A be a set of k (≥ 5) integers such that $|A + 2 \cdot A| = 3k$. Then $A = [0, k+1] \setminus \{x, y\}$, where $\{x, y\}$ is any one of the sets $\{0, 2\}$, $\{0, k\}$, $\{1, k+1\}$, and $\{k-1, k+1\}$.*

Theorem 1.5. *Let A be a set of six integers such that $|A + 2 \cdot A| = 19$. Then $A = [0, k+2] \setminus \{x, y, z\}$, where $\{x, y, z\}$ is any one of the sets $\{1, 3, 5\}$, $\{3, 5, 7\}$, $\{1, 3, 7\}$, $\{1, 5, 7\}$, $\{x, 7, 8\}$, where $x \in [2, 4]$, and $\{0, 1, z\}$, where $z \in [4, 6]$.*

Theorem 1.6. *Let A be a set of k (≥ 7) integers such that $|A + 2 \cdot A| = 3k + 1$. Then $A = [0, k+2] \setminus \{x, y, z\}$, where $\{x, y, z\}$ is any one of the sets $\{0, 1, z\}$, where $z \in [4, k]$, and $\{x, k+1, k+2\}$, where $x \in [2, k-2]$.*

The second part of this article is based on the connection between the sum of dilates and the Baumslag–Solitar group. To see this connection one can see the work of Freiman et al. [5,6]. For integers m and n, the general *Baumslag–Solitar group*, $BS(m, n)$, is a 1-relator group with two generators a and b with the defining relation $a^m b = b a^n$, i.e.,
$$BS(m, n) := \langle a, b : a^m b = b a^n \rangle.$$

The earliest known reference on the Baumslag–Solitar group is documented in [1], though there may be more ancient references. The 1-relator groups play a significant role in geometric and combinatorial group theory by providing many explicit examples of finitely presented groups and the Baumslag–Solitar groups are no exception.

Different choices of the pair (m, n) produces radically different properties. For example, $BS(m, n)$ is solvable if $|m| = 1$ and for $|m| > 1$ and $|n| > 1$, the group becomes non-solvable. Also, $BS(m, n)$ is residually finite if and only if $|m| = |n|$ or when one of $|m|$ and $|n|$ equals

1. In all other cases, the group fails to be residually finite. In this article, we focus on the Baumslag–Solitar group

$$BS(1,2) = \langle a, b : ab = ba^2 \rangle,$$

which is the simplest Hopfian finitely presented group.

Let S and T be nonempty finite subsets of $BS(1,2)$ containing k_1 and k_2 elements, respectively. Also, let us assume that S is contained in the coset $b^r\langle a\rangle$ and T is contained in the coset $b^p\langle a\rangle$ for some natural numbers r and p. Then

$$S = \{b^r a^{x_0}, b^r a^{x_1}, \dots, b^r a^{x_{k_1-1}}\}$$

and

$$T = \{b^p a^{y_0}, b^p a^{y_1}, \dots, b^p a^{y_{k_2-1}}\},$$

where $A = \{x_0, x_1, \dots, x_{k_1-1}\}$ and $B = \{y_0, y_1, \dots, y_{k_2-1}\}$ are sets of integers. Clearly, $|S| = |A|$ and $|T| = |B|$. Since $ab = ba^2$, we get $a^x b^y = b^y a^{2^y x}$. The product set

$$ST = \{(b^r a^{x_i})(b^p a^{y_j}) : i \in \{0, 1, \dots, k_1 - 1\} \text{ and } j \in \{0, 1, \dots, k_2 - 1\}\}.$$

Now, $(b^r a^{x_i})(b^p a^{y_j}) = b^r(a^{x_i} b^p)a^{y_j} = b^r(b^p a^{2^p x_i})a^{y_j} = b^{r+p} a^{2^p x_i + y_j}$. Therefore,

$$|ST| = |2^p \cdot A + B|.$$

In particular, when $S = T$, we have $ST := S^2 = b^{2r} a^{2^r \cdot A + A}$. Moreover, if $r = 1$, then

$$|S^2| = |A + 2 \cdot A|.$$

Throughout this paper we use the following notation. For integers a and b and a set A of integers with $|A| \geq 1$,

$$a - A = \{a - a' : a' \in A\} \quad \text{and} \quad ba^A = \{ba^{a'} : a' \in A\}.$$

If $a \leq b$, then

$$[a, b] = \{a, a + 1, \dots, b\}.$$

2. Preliminaries and Proof technique of Theorem 1.2–1.6

We can derive the following theorem simply from Theorem 1.1 (ii).

Theorem 2.1. *Let $k \geq 3$ be an integer and A be a finite set of k integers such that* $\min(A) = 0$ *and $d(A) = 1$. Then we have the following statements.*

(a) *If $|A + 2 \cdot A| = 3k - 1$, then for $k \geq 4$, $A \subseteq [0, k]$.*
(b) *If $|A + 2 \cdot A| = 3k$, then for $k \geq 5$, $A \subseteq [0, k + 1]$.*
(c) *If $|A + 2 \cdot A| = 3k + 1$, then for $k \geq 6$, $A \subseteq [0, k + 2]$.*

Proposition 2.1. *Let A be a nonempty finite set of $k \geq 5$ integers such that $A = [0, k] \setminus \{x\}$, where $0 \leq x \leq k$. Then we have the following statements.*

(a) *If $x \in \{1, k - 1\}$, then $|A + 2 \cdot A| = 3k$.*
(b) *If $2 \leq x \leq k - 2$, then $|A + 2 \cdot A| = 3k + 1$.*
(c) *If $x \in \{0, k\}$, then $|A + 2 \cdot A| = 3k - 2$.*

Proof. (a) If $x = 1$, then $A = \{0\} \cup [2, k]$. Clearly,

$$A + 2 \cdot A = \{0\} \cup [2, 3k].$$

So,

$$|A + 2 \cdot A| = 3k.$$

If $x = k-1$, then $A = [0, k] \setminus \{k-1\} = k - \big(\{0\} \cup [2, k]\big)$, which is the translation of the set $\{0\} \cup [2, k]$. Therefore, $|A + 2 \cdot A| = 3k$.
(b) If $2 \leq x \leq k-2$, then $A = [0, x-1] \cup [x+1, k]$. Clearly,

$$A + 2 \cdot A = [0, 3k].$$

So, $|A + 2 \cdot A| = 3k + 1$.
(c) If $x \in \{0, k\}$, then A becomes an arithmetic progression. So, $|A + 2 \cdot A| = 3k - 2$.
This completes the proof of the proposition. □

Proof of Theorem 1.3. This follows directly from Theorem 2.1 (a) and Proposition 2.1.

Proposition 2.2. *Let A be a nonempty finite set of k (≥ 6) nonnegative integers such that $A = [0, k+1] \setminus \{x, y\}$, where $1 \leq x < y \leq k$. Then we have the following statements.*

(a) *If $\{x, y\}$ is one of the sets $\{1, 2\}$, $\{k-1, k\}$, $\{1, 3\}$, $\{k-2, k\}$, $\{1, 5\}$, $\{k-4, k\}$, or $\{1, k\}$, then $|A + 2 \cdot A| = 3k + 2$.*
(b) *If $\{x, y\}$ is one of the sets $\{1, 4\}$, $\{k-3, k\}$, $\{2, 4\}$, $\{k-3, k-1\}$ for $k \geq 7$, $\{1, y\}$ with $6 \leq y \leq k-1$, or $\{x, k\}$ with $2 \leq x \leq k-5$, then $|A + 2 \cdot A| = 3k + 3$.*
(c) *If $\{x, y\}$ is one of the sets*
  (i) $\{x, x+1\}$, *where* $2 \leq x \leq k-2$,
  (ii) $\{x, x+2\}$ *for* $k \geq 7$, *where* $3 \leq x \leq k-4$,
  (iii) $\{x, x+3\}$, *where* $2 \leq x \leq k-4$,
  (iv) $\{x, y\}$ *for* $k \geq 7$, *where* $y - x \geq 4$ *and* $2 \leq x \leq k-5$,
  then $|A + 2 \cdot A| = 3k + 4$.

Proof of the Proposition 2.2 can be done similarly as 2.1.

Remark 1. *In line with Proposition 2.2, we observe the following statements.*

(i) If $\{x, y\}$ is one of the sets $\{0, 1\}$, $\{0, k+1\}$, or $\{k, k+1\}$, then A becomes an arithmetic progression and hence, $|A + 2 \cdot A| = 3k - 2$.
(ii) If $\{x, y\}$ is $\{0, y\}$ with $2 \leq y \leq k$ or $\{x, k+1\}$ with $1 \leq x \leq k-1$, then $|A + 2 \cdot A|$ may be obtained from Proposition 2.1.

Lemma 2.2. *Let $k \geq 6$ and $A = [0, k+1] \setminus \{x, y\}$ with $0 \leq x < y \leq k+1$. Then $|A + 2 \cdot A| = 3k$ if and only if $\{x, y\}$ is one of the sets $\{0, 2\}$, $\{0, k\}$, $\{1, k+1\}$, or $\{k-1, k+1\}$.*

Proof. Let $P = \big\{(x, y) : 1 \leq x < y \leq k\big\}$. Consider the sets

$A_1 = \{1, 2\},\ A_2 = \{k-1, k\},\ A_3 = \{1, 3\},\ A_4 = \{k-2, k\},\ A_5 = \{1, 5\},\ A_6 = \{k-4, k\},$
$A_7 = \{1, k\},\ A_8 = \{1, 4\},\ A_9 = \{k-3, k\},\ A_{10} = \{2, 4\},\ A_{11} = \{k-3, k-1\},$
$A_{12} = \{1, y\}$, where $6 \leq y \leq k-1$, $A_{13} = \{x, k\}$, where $2 \leq x \leq k-5$,
$A_{14} = \{x, x+1\}$, where $2 \leq x \leq k-2$, $A_{15} = \{x, x+2\}$, where $3 \leq x \leq k-4$,
$A_{16} = \{x, x+3\}$, where $2 \leq x \leq k-4$,
$A_{17} = \{x, y\}$, where $(y - x) \geq 4$, $y \geq 6$, and $2 \leq x \leq k-5$.

Then we have

(i) for $k = 6$,

$$P \subseteq A_1 \cup A_3 \cup A_4 \cup A_5 \cup A_6 \cup A_7 \cup A_8 \cup A_{10} \cup A_{11} \cup A_{14} \cup A_{16};$$

(ii) for $k \geq 7$,

$$P \subseteq A_1 \cup A_2 \cup \cdots \cup A_{17}.$$

The rest of the proof follows from Proposition 2.1 and Proposition 2.2. □

Proof of Theorem 1.4. This follows from Theorem 2.1 (b) and Lemma 2.2. □

Theorem 1.5 and Theorem 1.6 can be proved similarly.

3. Extended Inverse Results for the Subsets of $b\langle a\rangle$ in $BS(1,2)$

In this section, we apply Theorems 1.2–1.6 to derive extended inverse theorems for the subsets of $b\langle a\rangle$ in $BS(1,2)$. Freiman et al. [5] proved the following extended inverse problem for the subsets of $b\langle a\rangle$ in $BS(1,2)$.

Theorem 3.1 ([5]). *Let $A \subseteq \mathbf{Z}$ be a finite set of integers with $|A| = k \geq 1$. If $S = ba^A$ is a finite subset of the group $BS(1,2)$, then $|S| = k$ and*

$$|S^2| \geq 3k - 2.$$

Moreover, if $k \geq 3$ and

$$|S^2| = 3k - 2 + h < 4|S| - 4,$$

then $h \geq 0$ and S is a subset of the geometric progression

$$ba^u, ba^{u+d}, ba^{u+2d}, \ldots, ba^{u+(k+h-1)d}$$

of size $k + h \leq 2k - 3$, where $u = min(A)$ and $d = d(A)$.

Theorem 3.1 along with Theorems 1.2–1.6 enable us to derive the following extended inverse results in the group $BS(1,2)$.

Theorem 3.2. *Let $A \subseteq \mathbf{Z}$ be a finite set of integers with $|A| = k \geq 5$. Let also $S = ba^A$ be a finite subset of the group $BS(1,2)$ and $|S^2| = |A + 2 \cdot A| = 3k - 1$. Then there does not exist any S such that $|S^2| = 3k - 1$.*

Proof. From Theorem 1.1 and Theorem 3.1, we get

$$S \subseteq \{b, ba, ba^2, ba^3, \ldots, ba^k\}.$$

By Theorem 1.3, we know that there does not exist any such A for which $|A + 2 \cdot A| = 3k - 1$ for $k \geq 5$. So, we do not have any set S such that $|S^2| = 3k - 1$. □

Theorem 3.3. *Let $A \subseteq \mathbf{Z}$ be a finite set of integers with $|A| = k \geq 5$. Let also $S = ba^A$ be a finite subset of the group $BS(1,2)$ and $|S^2| = |A + 2 \cdot A| = 3k$. Then S is one of the following sets:*

(i) $S = \{ba, ba^3, ba^4, \ldots, ba^{k+1}\}$,
(ii) $S = \{ba, ba^2, ba^3, \ldots, ba^{k-1}, ba^{k+1}\}$,
(iii) $S = \{b, ba^2, ba^3, \ldots, ba^k\}$,
(iv) $S = \{b, ba, ba^2, ba^3, \ldots, ba^{k-2}, ba^k\}$.

Theorem 3.4. *Let $A \subseteq \mathbf{Z}$ be a finite set of six integers. Let also $S = ba^A$ be a finite subset of the group $BS(1,2)$ and $|S^2| = |A + 2 \cdot A| = 3k + 1$. Then S is one of the following sets:*

(i) $S = \{b, ba^2, ba^4, ba^6, ba^7, ba^8\}$,
(ii) $S = \{b, ba, ba^2, ba^4, ba^6, ba^8\}$,
(iii) $S = \{b, ba^2, ba^4, ba^5, ba^6, ba^8\}$,
(iv) $S = \{b, ba^2, ba^3, ba^4, ba^6, ba^8\}$,
(v) $S = \{b, \ldots, ba^{x-1}, ba^{x+1}, \ldots, ba^6\}$, *where* $x \in [2, 4]$,
(vi) $S = \{b, ba, ba^2, \ldots, ba^{z-1}, ba^{z+1}, \ldots, ba^7, ba^8\}$, *where* $z \in [4, 6]$.

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